MODULE – V

ENERGY THEOREMS IN ELASTICITY

TORSION OF NON CIRCULAR SHAFT

STRAIN ENERGY OF DEFORMATION:

Betti Rayleigh Reciprocal Theorem:

The forces of the first system (F_1, F_2, F_3, ... F_n) acting through the corresponding displacements produced by the second system do the same amount of work as done by a second system of forces (F'_1, F'_2, F'_3, ... F'_n) acting through the corresponding displacement produced by the first system of forces

STRAIN ENERGY OF DEFORMATION:

Proof:

Consider two system of forces F_1, F_2, F_3, ... F_n and F'_1, F'_2, F'_3, ... F'_n acting on a linear elastic body. Both the systems have the same point of application and the same directions. Let δ_1, δ_2, δ_3, ….. δ_3 be the corresponding displacements caused by F_1, F_2, F_3, ….. F_n and δ_1, δ_2, δ_3, …..δ_n be the corresponding displacements caused by F'_1, F'_2, F'_3, ... F'_n

$$F'_1\,\delta_1 + F'_2\,\delta_2 + F'_3\,\delta_3 + \ldots\ldots F'_n\,\delta_n = F'_1\,(a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + \ldots\ldots\ldots a_{1n}F_n) + F'_2\,(a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + \ldots\ldots\ldots a_{2n}F_n) + F'_3\,(a_{31}F_1 + a_{32}F_2 + a_{33}F_3 + \ldots\ldots\ldots a_{3n}F_n) + \ldots + F'_n\,(a_{n1}F_1 + a_{n2}F_2 + a_{n3}F_3 + \ldots\ldots\ldots a_{nn}F_n)$$

STRAIN ENERGY OF DEFORMATION:

$$= a_{11}F_1\,F'_1 + a_{12}F_2F'_1 + a_{13}F_3\,F'_1 + \ldots\ldots\ldots a_{1n}F_nF'_1 + a_{21}F_1F'_2 + a_{22}F_2F'_2 + a_{23}F_3\,F'_2 + \ldots\ldots a_{2n}F_nF'_2 + a_{31}F_1F'_3 + a_{32}F_2\,F'_3 + a_{33}F_3\,F'_3 + \ldots\ldots\ldots\ldots a_{3n}F_nF'_3 + \ldots\ldots\ldots\ldots + a_{n1}F_1\,F'_n + a_{n2}F_2\,F'_n + a_{n3}F_3 + \ldots\ldots a_{nn}F_n\,F'_n$$

$$= a_{11}F_1\,F'_1 + a_{22}F_2F'_2 + a_{33}F_3\,F'_3 + \ldots\ldots + a_{nn}F_n\,F'_n + a_{12}\,(F_2F'_1 + F_1F'_2) + a_{13}(F_3\,F'_1 + F_1F'_3) + \ldots\ldots a_{1n}\,(F_1F'_n + F_nF'_1) + a_{23}\,(F_3\,F'_2 + F_2\,F'_3) + a_{24}(F_4F'_2 + F_2F'_4) + \ldots\ldots a_{2n}\,(F_nF'_2 + F_2F'_n) + \ldots\ldots + a_{n-1\,n}\,(F_nF'_{n-1} + F_{n-1}F'_n)$$

Using Maxwell's theorem, $a_{ij} = a_{ji}$

STRAIN ENERGY OF DEFORMATION:

$$F_1\,\delta'_1 + F_2\,\delta'_2 + F_3\,\delta'_3 + \ldots\ldots\ F_n\,\delta'_n = F_1\,(a_{11}F'_1 + a_{12}F'_2 + a_{13}F'_3 + \ldots\ldots\ldots a_{1n}F'_n)$$

$$+ F_2\,(a_{21}F'_1 + a_{22}F'_2 + a_{23}F'_3 + \ldots\ldots\ldots a_{2n}F'_n) +$$

$$F_3\,(a_{31}F'_1 + a_{32}F'_2 + a_{33}F'_3 + \ldots\ldots\ldots a_{3n}F'_n) + ..$$

$$+ F_n\,(a_{n1}F'_1 + a_{n2}F'_2 + a_{n3}F'_3 + \ldots\ldots\ldots a_{nn}F'_n)$$

$$= a_{11}F'_1\,F_1 + a_{12}F'_2\,F_1 + a_{13}F'_3\,F_1 + \ldots\ldots\ldots a_{1n}F'_nF_1 +$$

$$a_{21}F'_1\,F_2 + a_{22}F'_2\,F_2 + a_{23}F'_3\,F_2 + \ldots\ldots\ldots a_{2n}F'_n\,F_2 +$$

$$a_{31}F'_1\,F_3 + a_{32}F'_2\,F_3 + a_{33}F'_3\,F_3 + \ldots\ldots\ldots a_{3n}F'_n\,F_3 + ..$$

$$+ a_{n1}F'_1\,F_n + a_{n2}F'_2\,F_n + a_{n3}F'_3\,F_n + \ldots\ldots\ldots a_{nn}F'_n\,F_n$$

STRAIN ENERGY OF DEFORMATION:

$$= \quad a_{11}F_1\,F'_1 + a_{22}F_2F'_2 + a_{33}F_3\,F'_3 + \ldots\ldots\ + a_{nn}F_n\,F'_n +$$

$$a_{12}\,(F_2F'_1 + F_1F'_2) + a_{13}(F_3\,F'_1 + F_1F'_3) + \ldots\ldots\ a_{1n}\,(F_1F'_n + F_nF'_1)$$

$$a_{23}\,\,(F_3\,F'_2 + F_2\,F'_3) + a_{24}(F_4F'_2 + F_2F'_4) + \ldots\ldots\ a_{2n}\,(F_nF'_2 + F_2F'_n)$$

$$+ \ldots\ldots\ + a_{n-1\;n}\,(F_nF'_{n-1} + F_{n-1}F'_n)$$

$$\mathbf{F'_1\,\delta_1 + F'_2\,\delta_2 + F'_3\,\delta_3 + \ldots\ldots\ F'_n\,\delta_n = F_1\,\delta'_1 + F_2\,\delta'_2 + F_3\,\delta'_3 + \ldots\ldots\ F_n\,\delta'_n}$$

Thus the forces of the first system (F_1, F_2, F_3, ... F_n) acting through the corresponding displacements produced by the second system do the same amount of work as done by a second system of forces (F'_1, F'_2, F'_3, ... F'_n) acting through the corresponding displacement produced by the first system of forces

STRAIN ENERGY OF DEFORMATION:

FAQs:

1. Define Strain Energy and Complementary Strain Energy.
2. Derive expression for strain energy in case of: axial loading, shear stress, bending and torsion.
3. State and prove reciprocal theorems

ENERGY METHODS IN ELASTICITY:

Castiglano's First Theorem:

If the strain energy U of a structure is expressed as a function of generalized force F_i then, first partial derivative of U with respect any one of the generalised force F_i is equal to the corresponding displacement δ_i

$$\frac{\partial U}{\partial F_i} = \delta_i$$

ENERGY METHODS IN ELASTICITY:

Proof:

Strain energy stored in an elastic body is given by

$$U = \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + F_3\delta_3 + \dots\dots.. + F_n\delta_n)$$

$$= \frac{1}{2}F_1(a_{11}F_1 + a_{12}F_2 + a_{13}F_3 + \dots\dots.. + a_{1n}F_n)$$

$$+\frac{1}{2}F_2(a_{21}F_1 + a_{22}F_2 + a_{23}F_3 + \dots\dots.. + a_{2n}F_n)$$

$$+\frac{1}{2}F_3(a_{31}F_1 + a_{32}F_2 + a_{33}F_3 + \dots\dots.. + a_{3n}F_n)$$

$$+\frac{1}{2}F_n(a_{n1}F_1 + a_{n2}F_2 + a_{n3}F_3 + \dots\dots.. + a_{nn}F_n)$$

ENERGY METHODS IN ELASTICITY:

$$U = \frac{1}{2}(a_{11}{F_1}^2 + a_{22}{F_2}^2 + a_{33}{F_3}^2 + \dots\dots.. + a_{nn}{F_n}^2)$$

$$+a_{12}F_1F_2 + a_{13}F_1F_3 + \dots\dots.. + a_{1n}F_1F_n$$

$$+a_{23}F_2F_3 + a_{24}F_2F_4 + \dots\dots.. + a_{2n}F_2F_n$$

$$\dots\dots. + a_{(n-1)n}F_{(n-1)}F_n$$

In the above expressions F_1, F_2, … are the generalized forces i.e., concentrated loads, moment or torques.

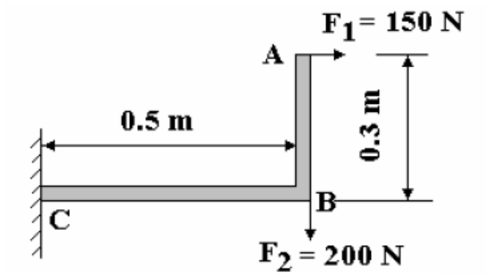
a_{11}, a_{12}, a_{13},…… are influence coefficients.

$$\frac{\partial U}{\partial F_1} = a_{11}F_1 + a_{12}F_2 + a_{13}F_1 + \cdots . + a_{1n}F_n$$

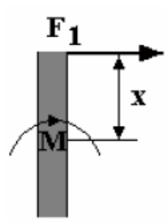
$$\frac{\partial U}{\partial F_1} = \delta_1 \qquad \frac{\partial U}{\partial F_2} = \delta_2 \qquad \frac{\partial U}{\partial F_i} = \delta_i$$

ENERGY METHODS IN ELASTICITY:

The diagram shows a simple frame with two loads. Determine the deflection at both. The flexural stiffness of both sections is 2 MNm^2.



SECTION AB Measure the moment arm x from the free end.



$M = F_1\,x$ (x measured from the free end)

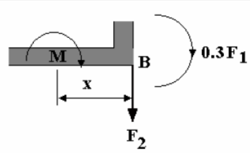
$$U = \frac{1}{2EI}\int_0^{0.3} M^2\,dx = \frac{1}{2EI}\int_0^{0.3} (F_1 x)^2\,dx = \frac{F_1^2}{2EI}\int_0^{0.3} x^2\,dx$$

$$U = \frac{F_1^2}{2EI}\left[\frac{x^3}{3}\right]_0^{0.3} = \frac{F_1^2}{2 \times 2 \times 10^6}\left[\frac{0.3^2}{3} - 0\right]$$

$$U = 2.25 \times 10^9\,F_1^2 \text{ Joules}$$

ENERGY METHODS IN ELASTICITY:

SECTION BC



$M = 0.3\,F_1 + F_2\,x$

$$U = \frac{1}{2EI}\int_0^{0.5} M^2\,dx = \frac{1}{2EI}\int_0^{0.5}(0.3F_1 + F_2 x)^2\,dx = \frac{1}{2EI}\int_0^{0.5}\left\{(0.3F_1)^2 + (F_2^2 x^2) + (0.6F_1F_2x)\right\}dx$$

$$U = \frac{1}{2EI}\int_0^{0.5}\left\{0.09F_1^2 x + F_2^2 x^2 + 0.6F_1F_2\right\}dx = \frac{1}{2EI}\left[0.09F_1^2 x + \frac{F_2^2 x^3}{3} + \frac{0.6F_1F_2x^2}{2}\right]_0^{0.5}$$

$$U = \frac{1}{2 \times 2 \times 10^6}\left[0.09F_1^2 + \frac{0.5^3 F_2^2}{3} + \frac{0.6F_1F_2 \times 0.5^2}{2}\right]$$

$U = 11.25F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75\,F_1F_2 \times 10^{-9}$

The total strain energy is

$U = 11.25F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75\,F_1F_2 \times 10^{-9} + 2.25 \times 10^{-9}\,F_1^2$

$U = 13.5F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75\,F_1F_2 \times 10^{-9}$

ENERGY METHODS IN ELASTICITY:

To find y_1 carry out partial differentiation with respect to F_1.

$y_1 = \delta U/\delta F_1 = 27F_1 \times 10^{-9} + 0 + 18.75\ F_2 \times 10^{-9}$

Insert the values of F_1 and F_2 and $y_1 = 7.8 \times 10^{-6}$ m

To find y_2 carry out partial differentiation with respect to F_2.

$y_2 = \delta U/\delta F_2 = 0 + 20.834F_2 \times 10^{-9} + 18.75\ F_1 \times 10^{-9}$

Insert the values of F_1 and F_2 and $y_2 = 7 \times 10^{-6}$ m

ENERGY METHODS IN ELASTICITY:

Stiffness Coefficient k_{ij}:

Stiffness Coefficient k_{ij} is defined as the force developed along F_i at i when a unit displacement δ_j is introduced keeping $\delta_i = 0$.

$$\mathbf{F_i = k_{i1}\delta_1 + k_{i2}\delta_2 + k_{i3}\delta_3 + \cdots + k_{in}\delta_n}$$

ENERGY METHODS IN ELASTICITY:

Castiglano's Second Theorem:

If the strain energy U of a structure is expressed as a function of generalized displacement δ_i then, first partial derivative of U with respect any one of the generalised displacement δ_i is equal to the corresponding generalised force F_i .

$$\frac{\partial U}{\partial \delta_i} = F_i$$

ENERGY METHODS IN ELASTICITY:

Proof:

Strain energy stored in an elastic body is given by

$$U = \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + F_3\delta_3 + \dots\dots.. + F_n\delta_n)$$

$$\begin{aligned} U &= \frac{1}{2}(F_1\delta_1 + F_2\delta_2 + F_3\delta_3 + \dots\dots.. + F_n\delta_n) \\ &= \frac{1}{2}\delta_1(k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + \dots\dots.. + k_{1n}\delta_n) \\ &\quad + \frac{1}{2}\delta_2(k_{21}\delta_1 + k_{22}\delta_2 + k_{23}\delta_3 + \dots\dots.. + k_{2n}\delta_n) \\ &\quad + \frac{1}{2}\delta_3(k_{31}\delta_1 + k_{32}\delta_2 + k_{33}\delta_3 + \dots\dots.. + k_{3n}\delta_n) \\ &\quad \dots + \frac{1}{2}\delta_n(k_{n1}\delta_1 + k_{n2}\delta_2 + k_{n3}\delta_3 + \dots\dots.. + k_{nn}\delta_n) \end{aligned}$$

ENERGY METHODS IN ELASTICITY:

$$U = \frac{1}{2}\left(k_{11}\delta_1{}^2 + k_{22}\delta_2{}^2 + k_{33}\delta_3{}^2 + \dots\dots .. + k_{nn}\delta_n{}^2\right)$$
$$+ k_{12}\delta_1\delta_2 + k_{13}\delta_1\delta_3 + \dots\dots .. + a_{1n}\delta_1\delta_n$$
$$+ k_{23}\delta_2\delta_3 + k_{24}\delta_2\delta_4 + \dots\dots .. + a_{2n}\delta_2\delta_n$$
$$\dots .... + a_{(n-1)n}\delta_{(n-1)}\delta_n$$

In the above expressions δ_1, δ_2, … are the generalized displacements i.e., translations or rotations.

k_{11}, k_{12}, k_{13},…… are influence coefficients.

$$\frac{\partial U}{\partial \delta_1} = k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + \cdots . + k_{1n}\delta_n$$

$$\frac{\partial U}{\partial \delta_1} = F_1 \qquad \frac{\partial U}{\partial \delta_2} = F_2 \qquad \frac{\partial U}{\partial \delta_i} = F_i$$

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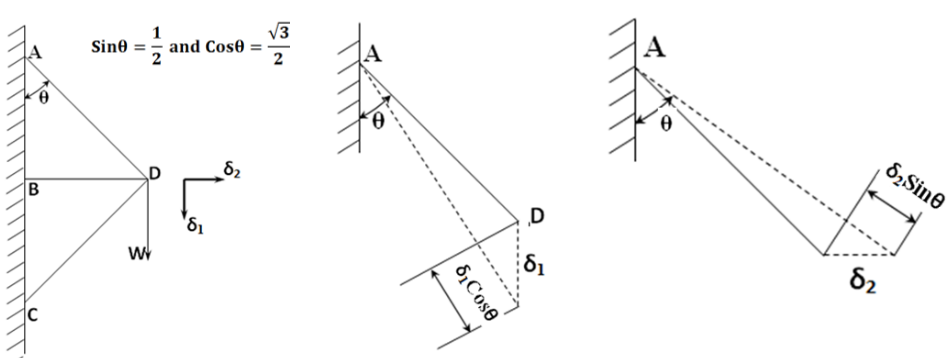
ENERGY METHODS IN ELASTICITY:

Problem:

Three elastic members AD, BD and CD are connected by smooth pins as shown in fig. All the members have same cross sectional area and are of same material. BD is 100 cms long and members AD and CD are each 200 cms long. What is the deflection under load W.



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ENERGY METHODS IN ELASTICITY:

Due to δ1,

BD will not undergo any change in length.

AD will extend by $\frac{\sqrt{3}}{2}\delta_1$

CD will be compressed by $\frac{\sqrt{3}}{2}\delta_1$

Due to δ_2,

BD will undergo extension by δ_2.

AD and CD will get extended by $\frac{1}{2}\delta_2$

Total extension of AD, $\boldsymbol{\delta_{AD}} = \frac{\sqrt{3}}{2}\boldsymbol{\delta_1} + \frac{1}{2}\boldsymbol{\delta_2}$

Total extension of BD, $\boldsymbol{\delta_{BD}} = \boldsymbol{\delta_2}$

ENERGY METHODS IN ELASTICITY:

Total extension of CD, $\boldsymbol{\delta_{CD}} = -\frac{\sqrt{3}}{2}\boldsymbol{\delta_1} + \frac{1}{2}\boldsymbol{\delta_2}$

If ‘a’ is the cross sectional area of the members,

Stresses on each member are:

$$\frac{\sigma}{\varepsilon} = E$$

$$\sigma_{AD} = \frac{E}{200}\left(\frac{\sqrt{3}}{2}\delta_1 + \frac{\delta_2}{2}\right)$$

$$\sigma_{BD} = \frac{E}{100}(\delta_2)$$

$$\sigma_{CD} = \frac{E}{200}\left(-\frac{\sqrt{3}}{2}\delta_1 + \frac{\delta_2}{2}\right)$$

Total Strain Energy, $\mathbf{U = U_{AD} + U_{BD} + U_{CD}}$ *Strain energy*, $U = \frac{\sigma^2}{2E} \text{ x aL}$

$$U = \frac{E^2}{200^2}\left(\frac{\sqrt{3}}{2}\delta_1 + \frac{\delta_2}{2}\right)^2 \text{x} \frac{\text{a x 200}}{2E} + \frac{E^2}{100^2}(\delta_2)^2 \text{x} \frac{\text{a x 100}}{2E} +$$

$$\frac{E^2}{200^2}\left(-\frac{\sqrt{3}}{2}\delta_1 + \frac{\delta_2}{2}\right)^2 \text{x} \frac{\text{a x 200}}{2E}$$

ENERGY METHODS IN ELASTICITY:

$$= \frac{aE}{1600}\left[6\delta_1^2 + 10\delta_2^2\right]$$

According to Castiglano's 2nd theorem,

$$\frac{\partial U}{\partial \delta_1} = W \qquad \frac{3aE}{400}\delta_1 = W \qquad \delta_1 = \frac{400W}{3aE}$$

$$\frac{\partial U}{\partial \delta_2} = 0 \qquad \frac{aE}{80}\delta_2 = 0 \qquad \delta_2 = 0$$

ENERGY METHODS IN ELASTICITY:

Principle of Virtual Work:

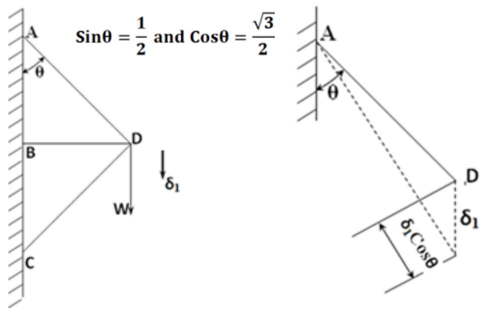
If a structure is in equilibrium and remains in equilibrium, while it is subjected to a virtual distortion, the external virtual work done δW is equal to the internal virtual work δU done by the internal stresses.

The virtual distortion given must be satisfying the constraint conditions i.e., the displacement should satisfy the displacement boundary condition.

ENERGY METHODS IN ELASTICITY:

Problem:

Solve the above problem using principle of Virtual Work.



Consider a virtual displacement δ_1 along the direction of applied force W.

Stresses on various members are

$$\sigma_{AD} = \frac{E}{200}\frac{\sqrt{3}}{2}\delta_1 \qquad \sigma_{BD} = 0 \qquad \sigma_{CD} = \frac{E}{200}\frac{\sqrt{3}}{2}\delta_1$$

ENERGY METHODS IN ELASTICITY:

Workdone by $\sigma_{AD} = \sigma_{AD}.\varepsilon_{AD}$ x Volume $= \frac{E}{200}\frac{\sqrt{3}}{2}\delta_1 \text{ x } \frac{\frac{\sqrt{3}}{2}\delta_1}{L} \text{ x A.L } = \frac{3AE}{800}{\delta_1}^2$

Workdone by $\sigma_{CD} = \sigma_{CD}.\varepsilon_{CD}$ x Volume $= \frac{E}{200}\frac{\sqrt{3}}{2}\delta_1 \text{ x } \frac{\frac{\sqrt{3}}{2}\delta_1}{L} \text{ x A.L } = \frac{3AE}{800}{\delta_1}^2$

Total internal virtual workdone by internal stresses, $\delta U = \frac{3AE}{800}{\delta_1}^2 + \frac{3AE}{800}{\delta_1}^2$

$$\delta U = \frac{3AE}{400}{\delta_1}^2$$

External Virtual Workdone, $\delta W = W\boldsymbol{\delta_1}$

$$\frac{3AE}{400}{\delta_1}^2 = W\boldsymbol{\delta_1} \qquad \delta_1 = \frac{400W}{3aE}$$

ENERGY METHODS IN ELASTICITY:

Principle of Minimum Potential Energy:

Statement: Of all the displacement fields which satisfies the prescribed constraint conditions, the correct state is that which makes the total potential energy of the structure a minimum.

Total Potential Energy, $\Pi = U + V - W_C$

Where, U – Elastic Energy stored in the deformed structure.

V – Negative of work done by external forces.

W_C – Work done by conservative forces.

When W_C =0, according to principle of minimum potential energy,

$$\delta\Pi = \delta(U + V) = 0$$

ENERGY METHODS IN ELASTICITY:

If Π is a function of δ_1, δ_2, δ_3 ..... δ_n then,

$$\delta\Pi = \frac{\partial\Pi}{\partial q_1}\delta q_1 + \frac{\partial\Pi}{\partial q_2}\delta q_2 \dots + \frac{\partial\Pi}{\partial q_n}\delta q_n$$

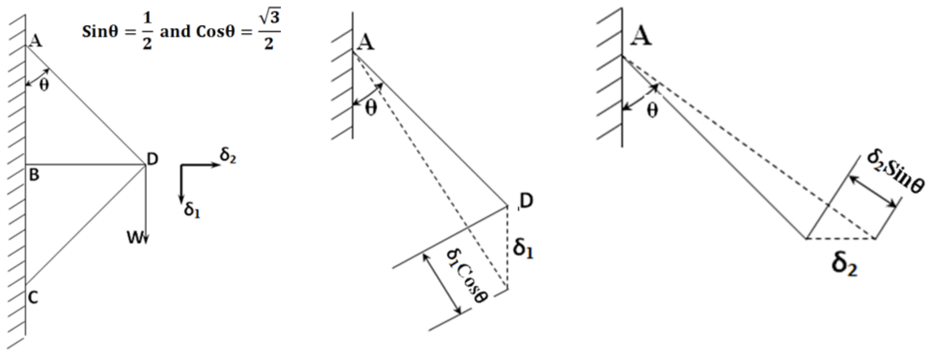
δΠ = 0 gives, $\frac{\partial\Pi}{\partial q_1}\delta q_1 = 0$, $\frac{\partial\Pi}{\partial q_2}\delta q_2 = 0$ .... $\frac{\partial\Pi}{\partial q_n}\delta q_n = 0$

Constraint conditions means displacements that can satisfy the displacement boundary conditions.

ENERGY METHODS IN ELASTICITY:

Problem:

Solve the above problem using principle of Virtual Work.



Let the displacements along vertical and horizontal directions be δ_1 and δ_2

Total Strain Energy $= \frac{aE}{1600}\left[6\delta_1{}^2 + 10\delta_2{}^2\right]$

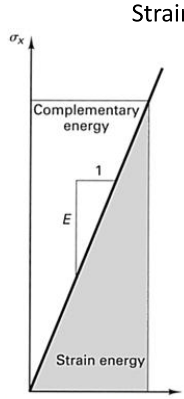
ENERGY METHODS IN ELASTICITY:

Principle of Minimum Complementary Energy:

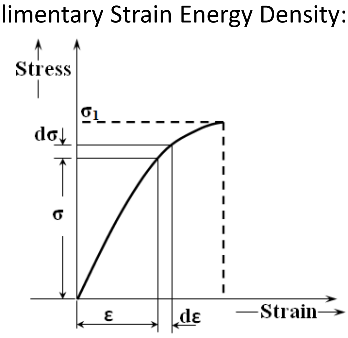
Of all the stress state which satisfies equations of equilibrium, the correct state is that which makes the total complementary energy of the structure a minimum.

Strain Energy density: $U = \int_0^{\sigma_1} \varepsilon d\sigma$

Complimentary Strain Energy Density: $U^* = \int_0^{\varepsilon_1} \sigma d\varepsilon$



Stress Strain Curve for linear material



Stress Strain Curve for non linear material

ENERGY METHODS IN ELASTICITY:

FAQs:

1. Define Strain Energy and Complementary Strain Energy.
2. Derive expression for strain energy in case of: axial loading, shear stress, bending and torsion.
3. State the following theorems: Reciprocal theorems, Castiglanos Theorems, Principle of Virtual Work, Principle of Minimum Potential Energy & Principle of minimum complementary strain energy
4. State and prove reciprocal theorems
5. State and prove Castiglanos theorems.

TORSION OF NON CIRCULAR SHAFTS

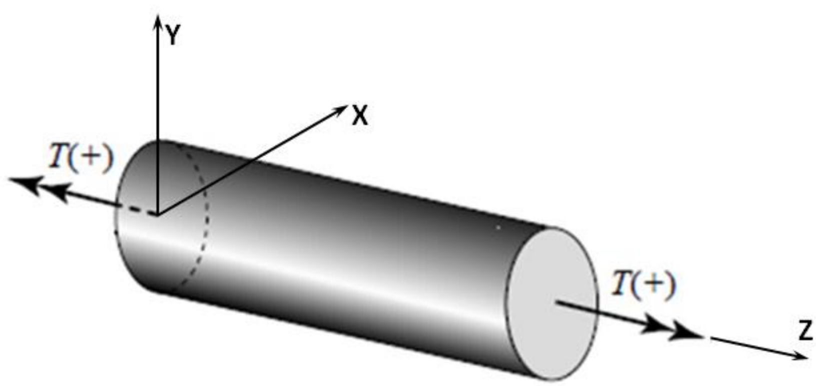
ST. VENANT'S METHOD

TORSION OF NON-CIRCULAR BARS:

Torsion of non-circular bars:

1. Saint Venant's theory - solutions for circular and elliptical cross-sections
2. Prandtle's method - membrane analogy
3. Torsion of thin walled open and closed sections- shear flow

TORSION OF NON-CIRCULAR BARS:



A circular bar subjected to Torque

TORSION OF NON-CIRCULAR BARS:

Torsion of Circular Shafts:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

T – Applied Torque in N-m.

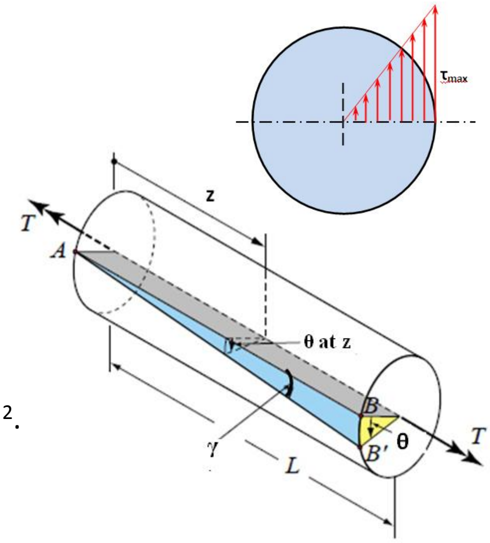
J – Polar Moment of Inertia m^4.

τ – Shear Stress at a radius r in N/m^2.

G – Modulus of Rigidity. in N/m^2.

θ – Angular Twist in Radians.

l – length considered in m.



TORSION OF NON-CIRCULAR BARS:

Torsion of Circular Shafts:

Assumptions:

1. The materiel is homogenous i.e of uniform elastic properties exists throughout the material.
2. The material is elastic, follows Hook's law, with shear stress proportional to shear strain.
3. The stress does not exceed the elastic limit.
4. **The circular section remains circular**
5. **Cross section remain plane.**
6. **Cross section rotate as if rigid i.e. every diameter rotates through the same angle**.

TORSION OF NON-CIRCULAR BARS:

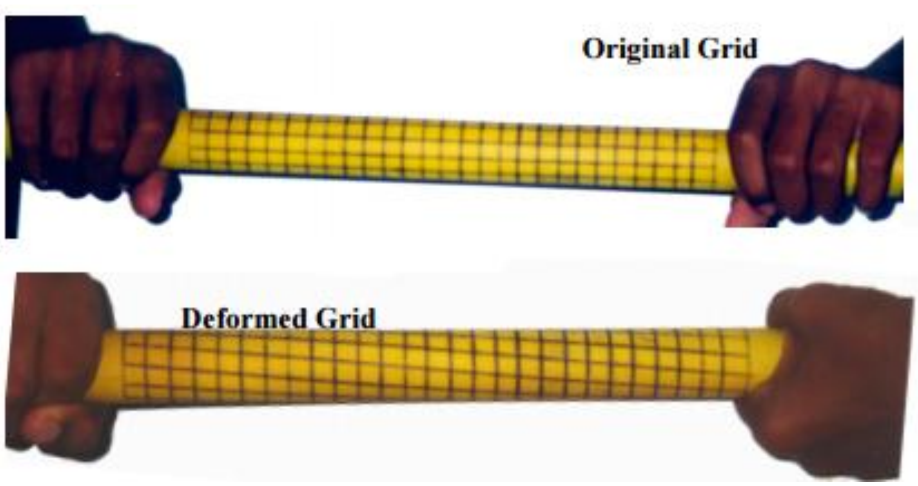
Torsion of Non - Circular Shafts:

In the case of circular shafts subjected to torsion, the circular section remains circular and Cross section remain plane. Cross section rotate as if rigid i.e. every diameter rotates through the same angle. Any point in the cross section will move along the x and y direction but not along z direction.

In the case of non – circular prismatic bars subjected to torsion, the points in the cross section will get displaced along x, y and z direction. This out of plane displacement along the axial direction of the bar is called **warping**.

TORSION OF NON-CIRCULAR BARS:

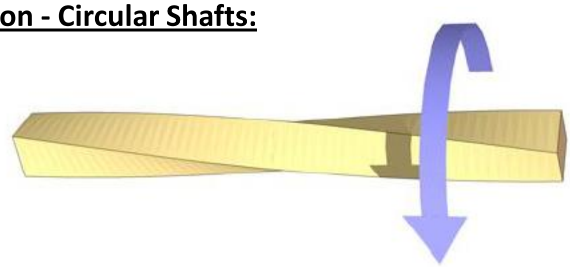
Torsion of Non - Circular Shafts:

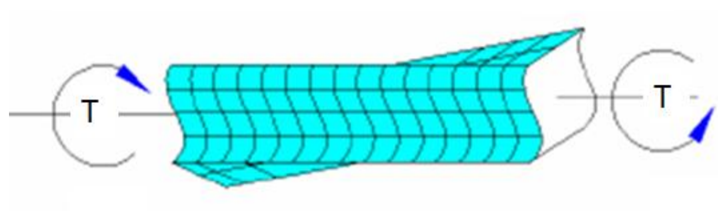


A circular bar subjected to Torque

TORSION OF NON-CIRCULAR BARS:

Torsion of Non - Circular Shafts:





A Non - circular bar subjected to Torque

TORSION OF NON-CIRCULAR BARS:

St. Venant's Inverse Method:

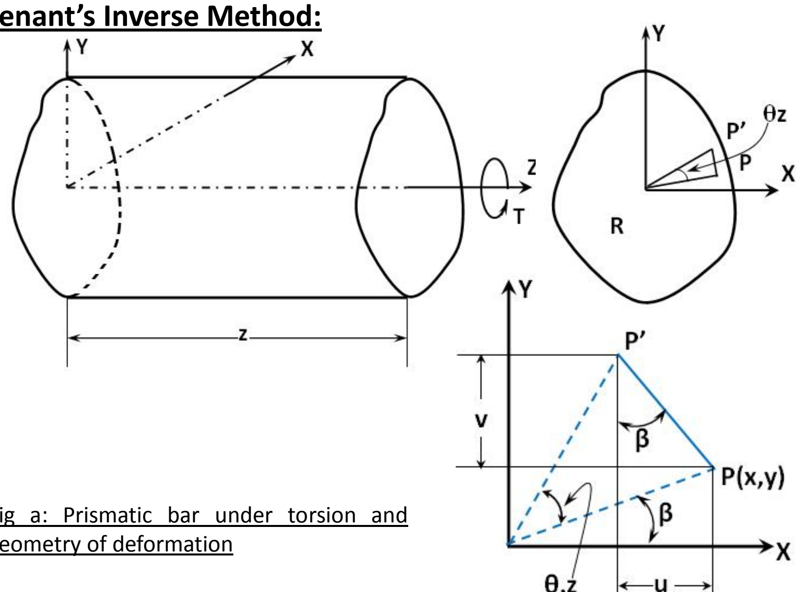


Fig a: Prismatic bar under torsion and geometry of deformation

TORSION OF NON-CIRCULAR BARS:

Consider the torsion of a prismatic bar of any cross section twisted by couples at the ends. The cross section . The cross section rotates about the axis and the twist per unit length is θ. Section at a distance z from the fixed end will rotate through an angle θ.z as shown in fig.a.

The displacement components along x & y directions are:

u = -r.θ.z.Sinβ and v = r.θ.z.Cosβ

where, $\sin\beta = \frac{y}{r}$ $\cos\beta = \frac{x}{r}$

TORSION OF NON-CIRCULAR BARS:

In addition to these x and y displacements the point P will undergo a displacement w in the z direction. This is called warping.

The z displacement is a function of x and y and is independent of z.

This means that warping is the same for all normal cross sections.

$$u = -\theta.y.z \quad \text{——} \quad (1) \qquad w = \theta.\psi(x,y) \quad \text{——} \quad (3)$$

$$v = \theta.x.z \quad \text{——} \quad (2)$$

θ* - is the angle twist angle at a length l from the fixed end.

TORSION OF NON-CIRCULAR BARS:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Substituting for u, v and w we get,

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

$$\gamma_{yz} = \theta\left(\frac{\partial \psi}{\partial y} + x\right) \qquad (4)$$

$$\gamma_{xz} = \theta\left(\frac{\partial \psi}{\partial x} - y\right) \qquad (5)$$

TORSION OF NON-CIRCULAR BARS:

From the Hooke's law,

$$\boldsymbol{\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0}$$

$$\boldsymbol{\tau_{yz} = G\gamma_{yz} = G\theta}\left(\frac{\partial \psi}{\partial y} + x\right) \qquad (6)$$

$$\boldsymbol{\tau_{xz} = G\gamma_{xz} = G\theta}\left(\frac{\partial \psi}{\partial x} - y\right) \qquad (7)$$

These components of stresses should follow the equilibrium equations.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

TORSION OF NON-CIRCULAR BARS:

The first two equations are identically satisfied. From the third equation we get.

$$\mathbf{G\theta}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad \text{———} \quad \text{I}$$

The warping function ψ satisfies the Laplace equation everywhere in the region R.

TORSION OF NON-CIRCULAR BARS:

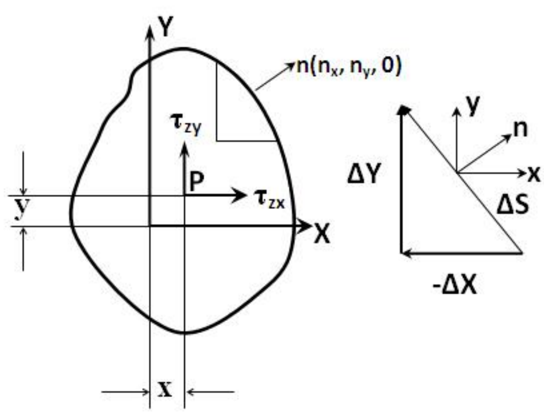


Fig (b): Cross section of the bar and the boundary conditions

TORSION OF NON-CIRCULAR BARS:

F_x, F_y anf F_z are the components of stress on a plane with outward normal n. At a point P on the surface the boundary condition has to be followed.

$$n_x\sigma_{xx} + n_y\tau_{xy} + n_z\tau_{xz} = F_x$$
$$n_x\tau_{xy} + n_y\sigma_{yy} + n_z\tau_{yz} = F_y$$
$$n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_{zz} = F_z$$

In this case, no forces act on the boundary,

i.e., Fx = Fy = Fz = 0

The first two equs. are identically satisfied and the third equ gives

$$G\theta\left(\frac{\partial\psi}{\partial x} - y\right)n_x + G\theta\left(\frac{\partial\psi}{\partial y} + x\right)n_y = 0$$

TORSION OF NON-CIRCULAR BARS:

From the figure (b) , Δs = ds; Δy = dy; Δx = dx.

$$n_x = \mathrm{Cos}(n, x) = \frac{dy}{ds}$$
$$n_y = \mathrm{Cos}(n, y) = -\frac{dx}{ds}$$

Substituting these values in the above equs.

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \quad \text{———} \quad \text{II}$$

Thus the torsion problem reduces to finding a function ψ which satisfies

1. equ I in the region R
2. equ II on the boundary S.

TORSION OF NON-CIRCULAR BARS:

The moment due to the stresses as given by equ 6 & 7 must be equal to the applied torque. The resultant forces in the x and y directions should vanish.

Referring to fig b, taking moments

Applied torque,

Substituting for the stresses from equ 6 and equ 7, we get

$$T = \iint_R (\tau_{yz} x - \tau_{xz} y)\ dx.dy$$

TORSION OF NON-CIRCULAR BARS:

$$T = G\theta \iint_R \left(x^2 + y^2 + x.\frac{\partial\psi}{\partial y} - y.\frac{\partial\psi}{\partial x}\right)\ dx.dy$$

Let

$$J = \iint_R \left(x^2 + y^2 + x.\frac{\partial\psi}{\partial y} - y.\frac{\partial\psi}{\partial x}\right)\ dx.dy$$

$$T = G.J.\theta \qquad \text{III}$$

J is called St. Venant's Torsional Constant

GJ is called Torsional Rigidity.

TORSION OF NON-CIRCULAR BARS:

Resultant Forces in the x direction vanishes

$$\iint_R \tau_{zx}.dx.dy = G\theta \iint_R \left(\frac{\partial\psi}{\partial x} - y\right).dx.dy$$

$$\frac{\partial\psi}{\partial x} - y = \frac{\partial\psi}{\partial x} - y + x\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$$

$$= \frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y} + x\right)\right]$$

Substituting the above in equ a we get,

$$\iint_R \tau_{zx}.dx.dy = G\theta \iint_R \frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y} + x\right)\right].dx.dy$$

TORSION OF NON-CIRCULAR BARS:

Resultant Forces in the x direction vanishes.

$$\iint_R \tau_{zx}.dx.dy = G\theta \iint_R \left(\frac{\partial\psi}{\partial x} - y\right).dx.dy \qquad \text{(a)}$$

$$\frac{\partial\psi}{\partial x} - y = \frac{\partial\psi}{\partial x} - y + x\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)$$

$$= \frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y} + x\right)\right]$$

Substituting the above in equ (a) we get,

$$\iint_R \tau_{zx}.dx.dy = G\theta \iint_R \frac{\partial}{\partial x}\left[x\left(\frac{\partial\psi}{\partial x} - y\right)\right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\psi}{\partial y} + x\right)\right].dx.dy$$

TORSION OF NON-CIRCULAR BARS:

Using Gauss Theorem, the above surface integral can be converted into a line integral

$$\iint_R \tau_{zx}.dx.dy = G\theta \oint x\left\{\left(\frac{\partial\psi}{\partial x} - y\right) n_x + \left(\frac{\partial\psi}{\partial y} + x\right) n_y\right\} dxdy$$

The expression within the curly braces is equal to zero according to equ I. i.e.,

$$\iint_R \tau_{zx}.dx.dy = 0$$

Similarly we can prove that $\iint_R \tau_{zy}.dx.dy = 0$

$$\boxed{\iint_R \left(\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}\right) dxdy = \oint_S P(x,y)n_x + Q(x,y)n_y}$$

TORSION OF NON-CIRCULAR BARS:

Summary of St. Venants Method

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \text{I}$$

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \qquad \text{II}$$

$$J = \iint_R \left(x^2 + y^2 + x.\frac{\partial\psi}{\partial y} - y.\frac{\partial\psi}{\partial x}\right) dx.dy$$

$$T = G.J.\theta \qquad \text{III}$$

$$\boldsymbol{\tau_{yz} = G\gamma_{yz} = G\theta}\left(\frac{\partial\psi}{\partial y} + x\right) \qquad \boldsymbol{\tau_{xz} = G\gamma_{xz} = G\theta}\left(\frac{\partial\psi}{\partial x} - y\right)$$

$$u = -\theta.y.z \qquad v = \theta.x.z \qquad w = \theta.\psi(x,y)$$

TORSION OF NON-CIRCULAR BARS:

Torsion of Circular Bars

The simplest solution to Laplace equ. Is

$$\psi = \text{Constant}, C$$

The boundary condition equ. II becomes,

$$-y\frac{dy}{ds} - x\frac{dx}{ds} = 0$$

$$\frac{d}{ds}\left(\frac{x^2 + y^2}{2}\right) = 0$$

$$x^2 + y^2 = \text{Constant}$$

Where x and y are the coordinates at any point in the boundary. Hence the boundary is a circle.

TORSION OF NON-CIRCULAR BARS:

From equ. III,

$$J = \iint_R (x^2 + y^2)\, dxdy$$

The above integral is the polar moment of inertia of an area bounded by a circle.

Hence, $T = GI_p\theta$

Where, I_P is the polar moment of inertia of the circle w.r.t its centre.

$$\theta = \frac{T}{GI_P}$$

$$w = \theta\psi \qquad w = \theta C = \frac{T.C}{GI_P}$$

TORSION OF NON-CIRCULAR BARS:

Since all the terms on the right hand side are constant for a given torque, given material and a given cross section. w is a constant at all cross sections. Since the centre has zero w, the value of w at every point in the cross section is zero. Thus the cross section does not warp.

The shear stresses are given by

$$\tau_{yz} = G\theta x = \frac{Tx}{I_P}$$

$$\tau_{xz} = -G\theta y = -\frac{Ty}{I_P}$$

The resultant stress is given by $\tau^2 = {\tau_{xz}}^2 + {\tau_{yz}}^2$

TORSION OF NON-CIRCULAR BARS:

$$\boldsymbol{\tau^2 = \frac{T^2\,(x^2+y^2)}{{I_P}^2}}$$

$$\boldsymbol{(x^2 + y^2) = r^2}$$

i.e., $\tau = \frac{Tr}{I_P}$

Direction of the resultant stress is given by

$$\tan\alpha = \frac{\tau_{yz}}{\tau_{xz}} \qquad \tan\alpha = \frac{G\theta x}{-G\theta y}$$

$$\tan\alpha = -\frac{x}{y}$$

TORSION OF NON-CIRCULAR BARS:

Torsion of Bars with Elliptical Sections:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad \text{———} \quad \text{(I)}$$

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0 \quad \text{———} \quad \text{(II)}$$

$$T = G.J.\theta \quad \text{———} \quad \text{(III)}$$

$$J = \iint_R \left(x^2 + y^2 + x.\frac{\partial\psi}{\partial y} - y.\frac{\partial\psi}{\partial x}\right) dx.dy$$

TORSION OF NON-CIRCULAR BARS:

Let $\psi = Axy$ ——— (1)

This satisfies the Laplace equation.

$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

Substituting for ψ, the above equ. becomes

$$(Ay - y)\frac{dy}{ds} - (Ax + x)\frac{dx}{ds} = 0$$

$$y(A - 1)\frac{dy}{ds} - x(A + 1)\frac{dx}{ds} = 0$$

TORSION OF NON-CIRCULAR BARS:

$$(\mathbf{A}+\mathbf{1})\mathbf{2x}\frac{\mathbf{dx}}{\mathbf{ds}} - \mathbf{2y}\,(\mathbf{A}-\mathbf{1})\,\frac{\mathbf{dy}}{\mathbf{ds}} = \mathbf{0}$$

$$\frac{\mathbf{d}}{\mathbf{ds}}\left[(\mathbf{A}+\mathbf{1})\mathbf{x}^2 - (\mathbf{A}-\mathbf{1})\mathbf{y}^2\right] = \mathbf{0}$$

$$(\mathbf{A}+\mathbf{1})\mathbf{x}^2 - (\mathbf{A}-\mathbf{1})\mathbf{y}^2 = \mathbf{Constant} \quad \text{———} \quad (2)$$

The above equation is of the form

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1} \quad \text{———} \quad (3)$$

Comparing equs. 2 and 3 we get

TORSION OF NON-CIRCULAR BARS:

$$\frac{\mathbf{a}^2}{\mathbf{b}^2} = \frac{\mathbf{1-A}}{\mathbf{1+A}} \qquad \mathbf{A} = \frac{\mathbf{b}^2 - \mathbf{a}^2}{\mathbf{b}^2 + \mathbf{a}^2} \quad \text{———} \quad (4)$$

$$\boldsymbol{\psi} = \frac{\mathbf{b}^2 - \mathbf{a}^2}{\mathbf{b}^2 + \mathbf{a}^2}\,\mathbf{xy} \quad \text{———} \quad (5)$$

This represents the warping function for an elliptical bar with semi axis a & b under torsion.

$$\mathbf{J} = \iint_{\mathbf{R}} \left(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{Ax}^2 - \mathbf{Ay}^2\right)\mathbf{dxdy}$$

$$= (\mathbf{A}+\mathbf{1})\iint_{\mathbf{R}} \mathbf{x}^2\mathbf{dxdy} + (\mathbf{1}-\mathbf{A})\iint_{\mathbf{R}} \mathbf{y}^2\mathbf{dxdy}$$

$$= (\mathbf{A}+\mathbf{1})\mathbf{I}_{\mathbf{y}} + (\mathbf{1}-\mathbf{A})\mathbf{I}_{\mathbf{x}} \quad \text{———} \quad (6)$$

TORSION OF NON-CIRCULAR BARS:

For an Elliptical area,

$$\mathbf{I_y} = \frac{\pi a^3 b}{4} \qquad \mathbf{I_x} = \frac{\pi a b^3}{4}$$

Substituting for Ix, Iy and A in the expression for J we get,

$$J = \frac{\pi a^3 b^3}{a^2 + b^2} \quad \text{———} \quad (7)$$

Torque,

$$T = GJ\theta$$

$$T = G\theta \frac{\pi a^3 b^3}{a^2 + b^2}$$

$$\theta = \frac{T}{G} \frac{a^2 + b^2}{\pi a^3 b^3} \quad \text{———} \quad (8)$$

TORSION OF NON-CIRCULAR BARS:

$$\boldsymbol{\tau_{yz}} = \mathbf{G\theta} \left(\frac{\partial \psi}{\partial y} + \mathbf{x} \right)$$

Substituting for θ from equ 2 and ψ from equ. 5

$$\boldsymbol{\tau_{zy}} = \mathbf{T}.\frac{a^2 + b^2}{\pi a^3 b^3} \left(\frac{b^2 - a^2}{b^2 + a^2} + 1 \right) \mathbf{x}$$

$$\boldsymbol{\tau_{yz}} = \frac{2Tx}{\pi a^3 b} \quad \text{———} \quad (9)$$

$$\boldsymbol{\tau_{zx}} = \mathbf{G\theta} \left(\frac{\partial \psi}{\partial x} - \mathbf{y} \right)$$

$$\boldsymbol{\tau_{zx}} = \frac{2Ty}{\pi a b^3} \quad \text{———} \quad (10)$$

TORSION OF NON-CIRCULAR BARS:

Resultant Stress

$$\tau = \left[\tau_{zy}{}^2 + \tau_{zx}{}^2\right]^{1/2}$$

$$\tau = \frac{2T}{\pi a^3 b^3}\left[b^4x^2 + a^4y^2\right]^{1/2}$$

Expression for maximum Shear Stress:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

TORSION OF NON-CIRCULAR BARS:

substituting for x in the expression for τ we get

$$\tau = \frac{2T}{\pi a^3 b^3}\left[a^2b^4 + a^2\left(a^2 - b^2\right)y^2\right]^{1/2}$$

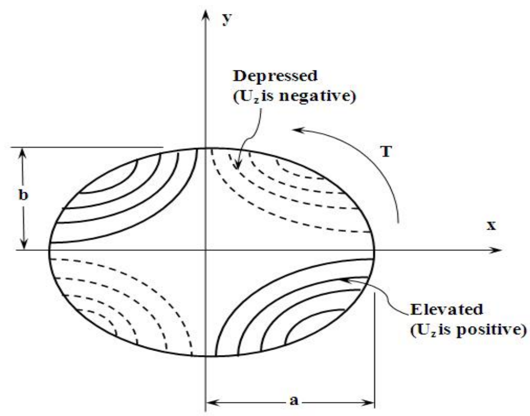
Since all the terms within the square brackets are positive, τ will be maximum when y is maximum i.e., when y =b.

Thus τ_{max} occurs at the ends of the minor axis

$$\tau_{max} = \frac{2T}{\pi a^3 b^3}\left(a^4b^2\right)^{1/2} = \frac{2T}{\pi a b^2}$$

Axial Displacement, $$w = \theta\psi = \frac{T\left(b^2 - a^2\right)}{\pi a^3 b^3 G}xy$$

TORSION OF NON-CIRCULAR BARS:



Cross section of an Elliptical bar showing Contour Lines of constant U_z

TORSION OF NON-CIRCULAR BARS:

Contour lines giving w constant are hyperbolas. If the ends are free to warp, there are no normal stresses.

If one end is fixed, the warping is prevented at that end and consequently stresses are induced. These normal stresses are called torsion induced warping stresses.